Sampling extremal trajectories for planar rigid bodies

Weifu Wang and Devin Balkcom

Abstract This paper presents an approach to finding the time-optimal trajectories for a simple rigid-body model of a mobile robot in an obstacle-free plane. Previous work has used Pontryagin's Principle to find strong necessary conditions on time-optimal trajectories of the rigid body; trajectories satisfying these conditions are called *extremal trajectories*. The main contribution of this paper is a method for sampling the extremal trajectories sufficiently densely to guarantee that for any pair of start and goal configurations, a trajectory can be found that (provably) approximately reaches the goal, approximately optimally; the quality of the approximation is only limited by the availability of computational resources.

1 Introduction

Consider a single rigid body in an otherwise empty plane. The body can translate in various directions that are described relative to a frame rigidly attached the body, or can rotate around various rotation centers whose locations are also fixed relative to the body frame. Let there also be some bounds on the velocities of these translations and rotations. This paper attacks the problem of finding the shortest or fastest trajectory to move the body from one configuration to another.

The system studied is quite specific, compared to systems for which general-purpose motion planning algorithms have been successfully applied, and the algorithm we present is much weaker than algorithms derived from exact analytical descriptions of time-optimal trajectories for specific systems (*e.g.* Dubins [4], Reeds-Shepp [16], and others [18, 17, 11, 2, 1, 20, 9]). However, the theorems and results in this paper show, perhaps for the first time, that *provably* optimal motion planning is computationally feasible for a somewhat-general model of mobile robots in the plane.

Motion planning algorithms that take either a sampling approach (*e.g.* search-based planners [3], RRT-based planners [12], probabilistic roadmaps [10]) or an

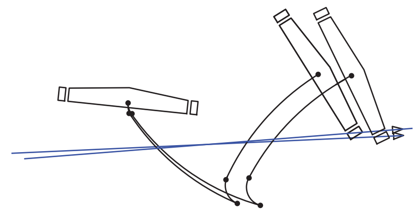


Fig. 1: Two control lines, and a corresponding extremal trajectory for each.

optimization approach [15, 14, 5] have been very effective in solving very generally-stated kino-dynamic motion-planning problems in the presence of obstacles. However, because these algorithms do not take advantage of the specific details of any particular robot model, these algorithms do not provably return trajectories that meet some optimality criteria, and in some cases, cannot guarantee that a solution that reaches the goal will be found, even if one exists.

Over the past few years, we have been working on generalizing the strong analytical results that have been found for various vehicles. In previous work, we have been able to show some strong geometric necessary conditions on optimal trajectories for a general model of planar rigid bodies [6]. This rigid body might represent a simple model of a mobile robot (for example, the steered cars studied by Dubins, Reeds and Shepp, and others [4, 16, 18, 11, 19], differential drives [2], omnidirectional vehicles [1, 20], or an object being manipulated in the plane by a robot arm, studied by *e.g.* Lynch [13]).

Essentially, every time-optimal trajectory can be characterized by a line in the plane, called the *control line*; motion of the body is essentially determined by the placement of this line, though the motion of the body is somewhat more complex than driving directly along the line. For example, figure 1 shows two characteristic control lines, and corresponding extremal trajectories; the vehicle is an asymmetric differential-drive robot – the maximum speed for the left wheel is 1.8 times the maximum speed for the right wheel. The problem of solving for an optimal trajectory would seem to be straightforward: given a starting configuration and a goal configuration, solve for the line placement that causes the vehicle to drive to the goal.

Solving for the placement of the control line is an inverse kinematics problem. For some simple robot geometries (*e.g.*, steered cars, differential drives, symmetric omnidirectional robots), an analytical solution can be found [4, 16, 2, 1, 20], but for other interesting variations of robot designs, we do not expect to be able to find analytical solutions.

An approach to solving an inverse kinematics problem is to sample a forwards kinematics problem. Sample the placements of the control line relative to the starting configuration, determine the structure of each trajectory, and choose a line whose corresponding trajectory reaches the goal. In [8], we tried this approach with apparently good results, but because for some configurations small changes in the placement of the line can dramatically change the resulting trajectory, we were unable to prove that the sampling of line placements could be made sufficiently dense to find trajectories close to the optimal.

The main contribution of this paper is an alternate sampling strategy, and a collection of theorems that prove that this sampling strategy is sufficient to find approximately time-optimal trajectories that approximately reach any goal configuration. By *approximately reach*, we mean that a trajectory is found that passes no more than some small distance ε from any particular configuration, where ε can be chosen as an arbitrarily small positive number. By *approximately optimal*, we mean that this trajectory has a time cost that is within δ of the optimal time cost, where δ can be chosen to be arbitrarily small, up to limits determined by computational precision and effort. Based on this collection of theorems, we sketch an algorithm that densely samples the space of trajectories, building a complete map of trajectories from a starting configuration (canonically chosen to be at the origin) to every other configuration that can be reached within some time bound $t_{\max}$.

The basic idea is that for every candidate trajectory structure, there is a segment whose duration is most sensitive to the location of the control line, called the *most-sensitive segment*. We sample the duration of this most-sensitive segment densely, and use this sampled duration to compute some constraints on the placement of the control line; specifically, the value of the Hamiltonian, H. This H value is characteristic of a trajectory, and if the identity of successive controls in the trajectory are known, fully determines the structure of the rest of the extremal trajectory. Further sampling the duration of the first control in the trajectory fully determines the placement of the control line.

2 Model and problem statement

We now state the problem and model more formally. Let there be a frame attached to the body. The configuration of the body is described by the position and orientation of this frame relative to some some world frame, $q = (x, y, \theta)$.

Let the control of the body be a Lebesgue-integrable vector function $u(t) \in \mathbf{R}^3$ that describes the translational and rotational velocities of the body at each time. For example, a constant function $u(t) = (1, 0, -1)$ would indicate that the body should translate with velocity one in the direction of the first axis of the body frame, and rotate with angular velocity -1 (a rotation in the clockwise direction).

The trajectory of the body in the world frame is determined by integrating the generalized velocity in the world frame. The controls can be transformed into velocities in the world frame by a 3x3 matrix R that is formed by replacing the upper left block of a 3x3 identity matrix with a 2x2 rotation matrix:

$$q(t) = q(0) + \int_0^t R(\theta(\tau))u(\tau) \qquad (1)$$

Let the vector function $u(t)$ be constrained within a polyhedron. Polyhedral control sets appear frequently in models of mobile robots or rigid bodies being pushed stably within the plane. For example, bounds on steering angle and translational velocity for a steered car, or bounds on wheel speeds for differential-drive or three-wheeled omnidirectional robot, lead to linear (and thus polyhedral) constraints on controls.

The formal problem statement is, given an pair of start and goal configurations for a rigid body, as well as the polyhedron bounding the velocity controls, find a time-optimal trajectory between the two configurations.

3 Necessary conditions for time optimality

In previous work, we have used Pontryagin's Maximum Principle to study necessary conditions on time-optimal trajectories for the rigid-body system. We call the trajectories satisfying the Maximum Principle *extremal*, and only a subset of extremal trajectories are optimal.

The Maximum Principle states that the Hamiltonian value of the system is the product of the control and an adjoint function, and during an optimal trajectory the control must maximize the Hamiltonian value, which is further a constant during this trajectory.

For the rigid body system we study, we have shown in previous work [7] that the Hamiltonian can be written in the following way:

$$H = k_1\dot{x} + k_2\dot{y} + \dot{\theta}(k_1 y - k_2 x + k_3), \tag{2}$$

where k_1, k_2 and k_3 are constants of integration, and $(\dot{x}, \dot{y}, \dot{\theta}) = R(\theta(t))u(t)$ almost everywhere. Along any optimal trajectory, the control $u(t)$ must maximize equation 2, for some choice of constants. We can thus think of the Hamiltonian maximization equation as a control law that describes the evolution of extremal trajectories.

Initial configuration of the body and the particular choice of constants essentially determine the structure of the trajectory. For example, if $k_1 = k_2 = 0$, and $k_3 > 0$, then the Hamiltonian reduces to $H = k_3\dot{\theta}$, and the controls must constantly maximize angular velocity. In previous work, we have labelled trajectories of this type *whirl*, and found a complete analytical solution method. Choosing different constants leads to different trajectories; the main problem is then to find values for the constants that generate a trajectory to a particular desired goal.

If $k_1^2 + k_2^2 \neq 0$, it is convenient to scale the constants so that $k_1^2 + k_2^2 = 1$. Then there is a nice geometric interpretation of the results of the Maximum Principle. For a particular choice of constants, we can draw a line in the plane with heading along the vector (k_1, k_2), a (signed) distance from the origin k_3. Changing the constants moves this *control line* and gives different trajectory structures (figure 1). If we attach a frame to the line, with first axis in the direction of (k_1, k_2) and second axis in the direction $(-k_2, k_1)$, then the Hamiltonian can be written more simply as:

$$H = \dot{x}_L + y_L \dot{\theta}, \tag{3}$$

with subscript L indicating coordinates measured in the control line frame. $\dot{x}_L$ is the projection of the translational component of the body velocity onto the control line, and y_L is the distance of the reference point on the robot to the control line.

For some choices of control line and starting configuration, the evolution of the trajectory is completely determined by the Maximum Principle and the control line; we call such trajectories *generic*, and they are the focus of this paper. Although there are non-generic (singular and whirl) trajectories, the problem of finding the fastest non-generic trajectory was previously solved analytically in [6].

3.1 Discrete control set

The maximization condition implies that only controls at vertices of the polyhedral control space occur in time-optimal generic trajectories [6]; thus we only need to consider a discrete control set. We denote the available controls (vertices of the original polyhedral control space) using $\mathbf{u}_i$, for $i \in \{1,2,\ldots,m\}$. For example, the control $\mathbf{u}_1 = (1,0,0)$, would correspond to driving forwards, and the control $\mathbf{u}_2 = (0,0,1)$ would correspond to spinning counterclockwise.

A trajectory structure is a sequence of controls. Denote the trajectory structure by $\mathbf{s}$. For example, $\mathbf{s} = (1,3,2,1,3,2,\ldots)$. The length of $\mathbf{s}$ is n, indicating the trajectory has n segments. It is often necessary to refer to the control at the ith segment in a trajectory. For convenience, we define a function $U(s_i)$, such that $U(s_i) = \mathbf{u}_{s_i}$.

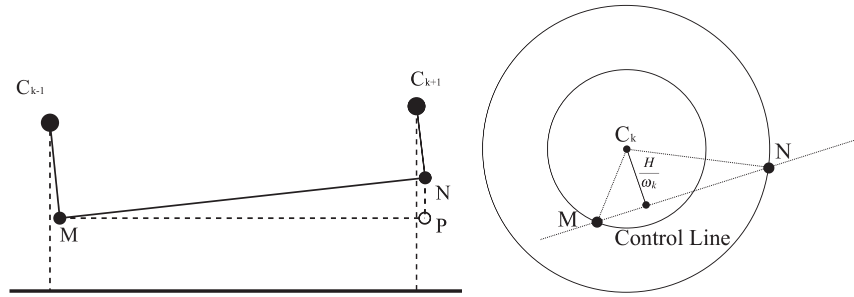
4 Sampling the space of control lines

This section presents the main results. Theorems 1 and 2 show that for a particular extremal trajectory structure, choosing the durations of the first segment and one other segment is sufficient to determine the placement of the control line, and thus the duration of every other segment except the last. Sampling these two durations sufficiently finely guarantees that extremal trajectories are sampled densely enough that a trajectory can be found that is in some sense *close enough* to the time-optimal in both time (theorem 3) and distance (theorem 4).

Although the paper is technically self-contained, proofs of the theorems rely strongly on geometric interpretations of the motion of the rigid body relative to the control line; these geometric interpretations are introduced more gently in [6].

Theorem 1 *Given the duration of the kth segment, t_k, and the identity of three consecutive controls, $U(s_{k-1})$, $U(s_k)$, $U(s_{k+1})$, where $k \in \{2,3,\ldots,n-1\}$, the Hamiltonian value H can be uniquely calculated.*

Proof. First consider the case where s_k is a translation control; let the velocity be v_k. Figure 2a shows an example. Let M be the location of the body reference point at



(a) Structure of kth segment where segment k is translation control

(b) Structure of kth segment where segment k is rotation control

Fig. 2: Figures for proof of theorem 1.

the switch *to* control $U(s_k)$, and let N be the location of this point at the switch *from* control $U(s_k)$. Let P be the point in the control-line frame with the same horizontal coordinate as N and the same vertical coordinate as M.

From equation 3, the magnitude of the projection of the translation velocity vector onto the control line is H. The magnitude of the velocity vector perpendicular to the control line is therefore $\sqrt{v_k^2 - H^2}$, and we can compute the length $\|N-P\|$:

$$\|N-P\| = t_k \sqrt{v_k^2 - H^2}. \tag{4}$$

We know the identities of the previous and next controls; we can use this and H to compute the distance of the rotation centers (C_{k-1} and C_{k+1}) to the control line. Specifically, C_{k-1} is a distance of H/ω_{k-1} from the control line, and C_{k+1} is H/ω_{k+1} from the control line. The radii of curvature corresponding to those controls are R_{k-1} and R_{k+1}.

The distance $\|N-P\|$ can also be determined from these two rotation controls, since the translation must be tangent to the circles centered at each rotation center with corresponding radii. We have:

$$\|N-P\| = |\left(\frac{H}{\omega_{k+1}} - \frac{HR_{k+1}}{v_k}\right) - \left(\frac{H}{\omega_{k-1}} - \frac{HR_{k-1}}{v_k}\right)|. \tag{5}$$

For notational brevity, denote $R_{k+1} - R_{k-1}$ by $R_{k+1,k-1}$, and $1/\omega_{k+1} - 1/\omega_{k-1}$ by $\omega_{k+1,k-1}$. Combining equations 4 and 5, and solving for H,

$$H = \frac{t_k v_k}{\sqrt{t_k^2 + \omega_{k+1,k-1}^2 + (\frac{R_{k+1,k-1}}{v_k})^2 - 2\omega_{k+1,k-1}\frac{R_{k+1,k-1}}{v}}}. \tag{6}$$

Now, consider the case where $U(s_k)$ is a rotation, shown in figure 2b. In previous work [6], we proved that for any switch from $U(s_m)$ to $U(s_n)$, there exists a point rigidly attached to the robot, such that at the switch between these controls, the point must be on the control line. Call switch $U(s_{k-1}) \rightarrow U(s_k)$ switch $k-1$, and call switch $U(s_k) \rightarrow U(s_{k+1})$ switch k.

Let M be the location of the switching point corresponding to the switch $k-1$, at the time of this switch. Let N be the location of the switching point corresponding to the switch k, at the time of this switch. The distances from C_k to N and from C_k to M are known [6]. The distance from C_k to the control line is H/ω_k.

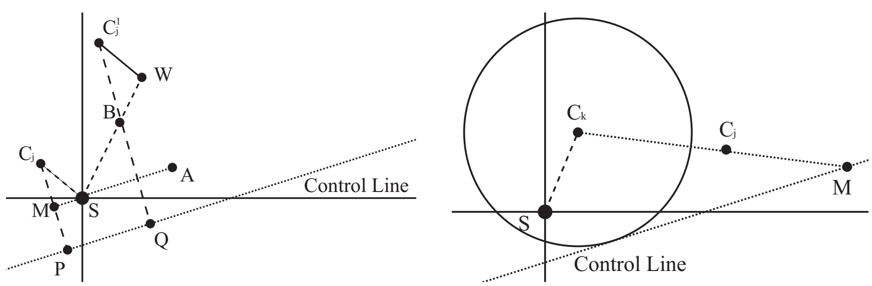
Some triangle geometry allows a solution for H. Specifically, denote the line passing through M and N by $y\cos\beta - x\sin\beta + c = 0$. Denote the coordinates of M by (x_1, y_1), the coordinates of N by (x_2, y_2), and the coordinates of C_k by (x_k, y_k). Then, we have:

$$y_1 \cos\beta - x_1 \sin\beta + c = 0 \tag{7}$$
$$y_2 \cos\beta - x_2 \sin\beta + c = 0 \tag{8}$$
$$\frac{H}{\omega_k} = |y_k \cos\beta - x_k \sin\beta + c| \ . \tag{9}$$

Solving the equation, we have:

$$\beta = \operatorname{atan2}(y_2 - y_1, x_2 - x_1) \tag{10}$$
$$c = y_1 \cos\beta - x_1 \sin\beta \tag{11}$$
$$H = \omega_k |y_k \cos\beta - x_k \sin\beta + c| \ . \tag{12}$$

□



(a) Case in which segment 1 is a translation (b) Case in which segment 1 is a rotation

Fig. 3: Figures for proof of theorem 2.

Theorem 2 *Given the identity of the first control, the Hamiltonian value, and the time for the first segment, t_1, the angle of the control line (α) and the distance to the origin (r) can be calculated; there may be zero, one, or two solutions.*

Proof. If $U(s_1)$ is translation, take figure 3a as an example. Denote the orientation of the control line by α. Denote the first control $U(s_1)$ by $(\dot{x}, \dot{y}, 0)$. Consider a unit vector **SA** pointing along $(\cos\alpha, \sin\alpha)$, and a unit vector **SB** pointing in the direction of $(\dot{x}, \dot{y})$, where $\dot{x}^2 + \dot{y}^2 = v_1^2$. Denote the angle between these two vectors by γ;

then $\cos\gamma = H/v_1$. Denote the location of the body reference point at the time of the first switch by W. We can then derive:

$$\mathbf{SB} = (\dot{x}, \dot{y});\ \ \mathbf{SA} = (\cos\alpha, \sin\alpha) \tag{13}$$
$$\|S - W\| = v_1 t_1;\ \ H/v_1 = (\dot{x}\cos\alpha + \dot{y}\sin\alpha)/v_1 \tag{14}$$
$$\tan^2\tfrac{\alpha}{2}(H + \dot{x}) - 2\dot{y}\tan\tfrac{\alpha}{2} + (H - \dot{x}) = 0 \tag{15}$$
$$\Rightarrow \tan\tfrac{\alpha}{2} = (\dot{y} + \sqrt{v_1^2 - H^2})/(H + \dot{x}). \tag{16}$$

There are up to two solutions for α. For each α, to determine r, find a control j with the largest distance to a line aligned with vector $(\cos\alpha, \sin\alpha)$. Denote that distance by d_j, the control j by $\mathbf{u}_j = (\dot{x}_j, \dot{y}_j, \dot{\theta}_j)$, the angular velocity by ω_j, where $\omega_j = \dot{\theta}_j$, and denote the radius by $R_j = \sqrt{(\frac{\dot{x}_j}{\dot{\theta}_j})^2 + (\frac{\dot{y}_j}{\dot{\theta}_j})^2}$. We know that after time t_1, that control will become the one maximizing the Hamiltonian value, so we have:

$$\|C_j - P\| = d_j;\quad \|M - P\| = r \tag{17}$$
$$d_j + t_1\sqrt{v^2 - H^2} = \tfrac{H}{\omega_j} = \|C_j^1 - Q\| \tag{18}$$
$$R_j \cos(\text{atan}\tfrac{\dot{y}_j}{\dot{x}_j} + \text{acos}\tfrac{H}{v_1}) + r = d_j. \tag{19}$$

If the first control is a rotation (figure 3b), we then know where the first rotation center C_1 is. At the same time, using C_1 and t_1, the duration of the first segment, we will know the configuration of the robot at the first switch.

For the first segment, we know C_1, and we know H. There exists a circle centered at C_1 with radius $\frac{H}{\omega_1}$. At the first switch, we can find all the switching points for switches from C_1 to every other control. The control line must pass a switching point M and tangent to the circle, as shown in figure 3b. At the same time, the next control $U(s_2)$ must satisfy the necessary conditions. Since this is not a singular trajectory, at each switch, there are at most two controls that could maximize the Hamiltonian value, so the next control is unique. Therefore, we can calculate the α and r.

Denote the first control $U(s_1)$ by $(\dot{x}_1, \dot{y}_1, \dot{\theta}_1)$. Consider the switching point for switch 1: $U(s_1)$ to $U(s_2)$ ($U(s_2)$ can be any possible control), denote this switching point by M. Since we know the first control and the duration of the first segment t_1, we can then calculate the coordinates of M at first switch, denote by (x_s, y_s). And we know C_1 has world frame coordinates $(-\frac{\dot{y}_1}{\dot{\theta}_1}, \frac{\dot{x}_1}{\dot{\theta}_1})$. Denote the control line by $y\cos\alpha + x\sin\alpha + r = 0$. Since the control line passes through (x_s, y_s) and has distance H/ω_1 to C_1, we have:

$$y_s\cos\alpha - y_s\sin\alpha + r = 0 \tag{20}$$
$$\tfrac{H}{\omega_1} = \tfrac{\dot{x}_1}{\dot{\theta}_1}\cos\alpha - \tfrac{-\dot{y}_1}{\dot{\theta}_1}\sin\alpha + r. \tag{21}$$

We can then test if the second control could maximize the Hamiltonian value, and discard any solutions that violate the Maximum Principle. Then, we can calculate α and r by solving the above system of equations. □

4.1 Approximation theorems

The two theorems stated above relate the position of the control line to the duration of different segments of an extremal trajectory. Assume we have two trajectories with the same structure (control sequence), starting at S. If the durations for all corresponding segments are similar, then we can see that at any time, the locations reached by the two trajectories will be close (as measured by Euclidean distance).

Lemma 1. *Consider two trajectories Y and Y' with identical structure, equal duration for all segments but one translation segment k, and a small distance κ. The corresponding end points are G and G'. For any translation control k in Y with duration t_k, and the same control k in Y_1 with duration t'_k, if $|t_k - t_k'| < \frac{\kappa}{v_k}$, $\|G - G'\| < \kappa$.*

Proof. Translation is commutative. □

Now, let us consider how changing the duration of a rotation control can change the end point. Given some small angle σ and a vector $\mathbf{v}$, assume we can rotate around any point on $\mathbf{v}$. We want to know how much the end point can move. Given two points on the vector P and Q, if P is further from the end point, then rotating around P can change the end point more than rotating around Q. In the following lemma, we will prove that the upper bound of the movement of the end point is $\mathbf{v}\sigma$.

Lemma 2. *Consider a trajectory Y with end point G, a small angle σ, and two points P and Q on the trajectory, with $\|P - G\| > \|Q - G\|$. Form a new trajectory Y_1 with end point G_1 by rotating the trajectory from P to G around P by δ, and form another trajectory Y_2 with end point G_2 by rotating the trajectory from Q to G by σ. Then, $\|G_2 - G\| < \|G_1 - G\|$. What is more, for any trajectory Y_k (end point G_k) achieved by rotating the trajectory Y around a series of points along the trajectory with σ angle in total, $\|G_k - G\|$ is upper bounded by only rotating σ around the furthest point from G on Y.*

Proof. The trajectory from P to G can be viewed as a vector pointing from P to G. So can the trajectory from Q to G. Rotating the trajectory from P to G is equivalent to rotating a vector **PQ** around P. Then

$$\|G_1 - G\| = 2\sin\frac{\sigma}{2}\|P - G\| \tag{22}$$

$$\|G_2 - G\| = 2\sin\frac{\sigma}{2}\|Q - G\| \tag{23}$$

Since $\|P - G\| > \|Q - G\|$, it follows that $\|G_2 - G\| < \|G_1 - G\|$.

Also, the combination of rotating around P and Q by a total angle of σ cannot move the end point further than $\|G_1 - G\|$. Denote $\|P - G\|$ by d_1, and denote $\|Q - G\|$ by d_2. Consider a rotation around P by σ_1, followed by a rotation around Q by $\sigma - \sigma_1$. Denote this new trajectory by Y_3, with end point G_3. By the triangle inequality $\|G_3 - G\| \leq 2\sin\frac{\sigma_1}{2} d_1 + 2\sin\frac{\sigma-\sigma_1}{2} d_2$. Let $d_2 = kd_1$ where $0 < k < 1$.

$$\sin\frac{\sigma_1}{2} + k\sin(\frac{\sigma-\sigma_1}{2}) \leq \sin\frac{\sigma}{2} \tag{24}$$

The upper bound is achieved when $\sigma_1 = \sigma$. Therefore, the change of the end point with any combination of rotations around points along trajectory with σ angle in total can be upper bounded by rotating around the furthest point from G with σ. □

Note that, for any given start and goal, we can find a upper time bound $t_{\max}$ for time-optimal trajectory by planning any (not necessarily optimal) path between the start and goal (Furtuna [6] gives a simple universal planner that can serve this purpose). At the same time, we can find a maximum speed over all controls. Denote the absolute value of this speed by $v^C_{\max}$ (compared to the fastest translation $v^T_{\max}$), and denote the absolute value of the fastest rotation by $\omega_{\max}$. Based on these two quantities, we can then calculate the furthest point from the goal the trajectory can reach (because the distance between the reference and the goal may not always be decreasing) $d_{\max} = t_{\max} v^C_{\max}/2$. Now, we will prove the following theorem.

Theorem 3 *Consider two trajectories Y' and Y with the same structure (control sequence), both starting at S and having time cost $t' = \sum_{i=1}^{n} t'_i$ and $t = \sum_{i=1}^{n} t_i$. Denote the point that Y' passes through at t' by G', and denote the point that Y passes through at t by G. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\sum_{i=1}^{n} |t'_i - t_i| < \delta$, $|G' - G| < \varepsilon$. Specifically, let δ be the minimum of $\frac{\varepsilon}{v^T_{\max}}$ and $\frac{\sqrt{(\omega_{\max} t_{\max} v^C_{\max}/2 + v^C_{\max})^2 + 2\omega_{\max} v^C_{\max}\varepsilon} - (\omega_{\max} t_{\max}/2+1) v^C_{\max}}{\omega_{\max} v^C_{\max}}$.*

Proof. Based on Lemma 1 and 2, we can derive an upper bound on how much a change in the duration of a segment can affect the location of the point a trajectory passes through at a certain time. The segment can either be a rotation or a translation. First let us consider translation. We have:

$$\|G^* - G\| < \varepsilon \tag{25}$$

$$\varepsilon \leq \delta v^T_{\max} \tag{26}$$

$$\delta \geq \frac{\varepsilon}{v^T_{\max}}. \tag{27}$$

Now consider the rotation case. According to Lemma 2, changing the duration of the rotation around the furthest point from G may lead to the biggest difference between trajectories in $\mathbf{R}^2$. At the same time, for a small time δ, any rotation cannot change the orientation of the body more than $\delta\omega_{\max}$. Then Y can be rotated at most $\delta\omega_{\max}$ from Y'. Denote the upper time bound for Y' by $t_{\max}$. The upper time bound for Y can then be represented by $t_{\max} + \delta$. Denote the furthest point by P for both trajectories by translating one of the trajectories such that the two trajectories overlap at P. Without loss of generality, $\|P-G'\| < \|P-G\|$. We have:

$$\|G'-G\| < \varepsilon \leq 2\sin(\tfrac{\omega_{\max}\delta}{2})\|P-G\| + \delta v^C_{\max} \tag{28}$$

$$\|P-G\| \leq \tfrac{v^C_{\max}(t+\delta)}{2} \tag{29}$$

$$\Rightarrow \varepsilon < \omega_{\max}\delta\,\tfrac{v^C_{\max}(t+\delta)}{2} + \delta v^C_{\max} \tag{30}$$

$$\Rightarrow \delta \geq \frac{\sqrt{(\omega_{\max}t_{\max}v^C_{\max}/2+v^C_{\max})^2+2\omega_{\max}v^C_{\max}\varepsilon}-(\omega_{\max}t_{\max}/2+1)v^C_{\max}}{\omega_{\max}v^C_{\max}}. \tag{31}$$

We require the difference between Y and Y' to be less than ε; therefore, choose the smaller δ. □

Finally, we need to prove that small changes in t_1 and t_k cannot change the duration of an extremal trajectory too much.

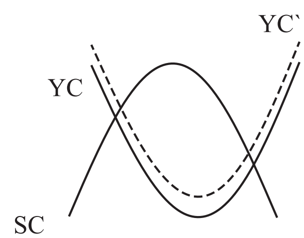


Fig. 4: The intersection between trajectories using the same controls (with different H values), with switching curves in $y-\theta$ space.

Theorem 4 *Consider two extremal trajectories Y and Y' with the same trajectory structure. Let the duration of the first segments be t_1 and t'_1. There exists a most-sensitive segment k, satisfying the properties below. Let the durations of segments k be t_k and t'_k. If $|t_1-t'_1| < \Delta t$ and $|t_k-t'_k| < \Delta t$, then $\sum_{i=1}^{n-1}|t'_i-t_i| < n\Delta t$*

Proof. Given a trajectory structure of one extremal trajectory, we will show that there exists a most sensitive translation control p and a most sensitive control q. A most sensitive segment k can be derived from control p and control q.

For the first $n-1$ segments, we want to prove that even if we change t_1 and t_k by Δt_1 and Δt, the total duration of those segments does not change more than $n\Delta t$. Then, we need to prove that for each segment, changing t_1 by Δt_1 and changing t_k by Δt does not change the duration by more than Δt.

The trajectory is represented by H and α. First let us consider α. As we have seen in the previous section, given H, changing α only affects the duration of the first segment. So, as long as we change the duration of the first control t_1 by $\Delta t_1 < \Delta t$, we can be assured that changing t_1 by Δt_1 will affect the total duration of the trajectory no more than Δt.

Now, let us consider the Hamiltonian value. For a small, fixed change of H, the duration of each segment $t_k, k = 2, 3, \ldots, n-1$ may change differently. We will prove that for this change ΔH, there exists a segment k such that the duration of kth segment (the most sensitive one) changes the most. We find the most sensitive segment k base on the most sensitive rotation control and the most sensitive translation control.

First, let us consider rotation. Consider the switching points in the control line frame. At each switch, we can observe the signed distance of the body from the control line y_L, and the angle of the body relative to the control line θ_L. Imagine fixing each switching point on the control line and rotating the robot around the switching point. We then get a set of sinusoids indicating the relative position of y_L and θ_L at a switch for different H values. If we plot these sinusoids in a frame where y_L is the second axis while θ_L is the first axis, we call these sinusoids *switching curves*, and this coordinate system $y-\theta$ *space*.

Now, consider generic trajectories in the $y-\theta$ space; take figure 4 as an example. Curves YC and YC' follow the same control sequences, but have different H values. Let the difference in H values be ΔH. To calculate how much the duration of this control changes, we need to calculate how much the θ changes at the intersection. For a unit change of θ, the y can change no more than max $\frac{dYC}{d\theta} * \Delta\theta$, which is denoted by Δy. We choose a suitable scale of θ and y such that the switching curve can be represented by $\sin\theta$ and the control curve TC (the control has angular velocity ω_i and the radius is R_i) is represented by $y_c = A\sin(\gamma\theta+\beta)+B$. We have the following

$$A = \frac{H}{\omega_i} \tag{32}$$

$$\Delta H \leq \Delta y + y'_c * \Delta\theta \tag{33}$$

$$\Delta y \leq \theta \tag{34}$$

$$\Delta\theta * y'_c \leq \Delta\theta\left(\frac{H}{\omega_k} - R_k\right) \tag{35}$$

$$\Delta t_k = \frac{2\Delta\theta}{\omega_k} \leq \frac{2\Delta H}{\omega_k + H} \leq \frac{2\Delta H}{\omega_k}. \tag{36}$$

So, the most sensitive rotation control (MSRC) is the one with the smallest absolute angular velocity. For unit change of duration of each segment, the ΔH for the most sensitive rotation control is the smallest. So, if we change the duration of the most sensitive control by Δt, the duration of all the other segments changes by less than Δt.

Now, let us consider the translation case. Two translations cannot be adjacent in an optimal trajectory [6]. So, a translation $U(s_k)$ (with speed v_k) must follow a rotation $U(s_{k-1})$. The translation is followed by a rotation $U(s_{k+1})$. Since we know the trajectory structure, we then can calculate the distance the translation need to cover in control line frame's y direction, take figure 2a as an example. We know $\|N-P\|$; therefore, we know the duration of the translation: $\frac{\|N-P\|}{\sqrt{v_k^2-H^2}}$. Then,

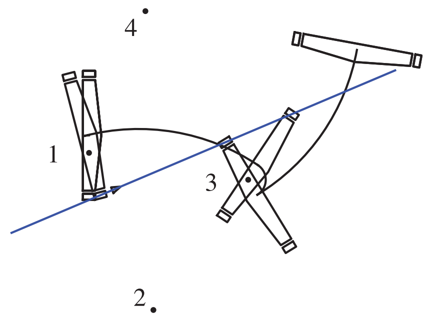


Fig. 5: An approximate time-optimal motion for the asymmetric differential-drive derived using the sampling algorithm, with a labelled sequence of rotation centers corresponding to each control.

$$\Delta t_k = |\omega_{k-1,k+1}|\left(\frac{H+\Delta H}{\sqrt{v_k^2-(H+\Delta H)^2}} - \frac{H}{\sqrt{v_k^2-H^2}}\right). \tag{37}$$

The translation with the smallest v_k is therefore the most sensitive translation control (MSTC). If we change the duration of that segment by Δt, then the ΔH is small enough to ensure that the duration change of all the other segments are smaller than Δt. Function $\frac{H+\Delta H}{\sqrt{v_k^2-(H+\Delta H)^2}} - \frac{H}{\sqrt{v_k^2-H^2}}$ is increasing with respect to H.

We can then compare the Δt_k calculated from rotation and translation to find the most sensitive segment k such that for any unit change of H, the duration of segment k changes the most. Therefore, if we change the duration of that segment by Δt, the duration of all the other segments will not change more than Δt, and the difference between t and t' is guaranteed to be bounded within $n\Delta t$. □

5 A sampling algorithm

Based on the above theorems, we now sketch an algorithm that can be used to sample the space of placements of the control line. The inputs to the sampling algorithm are: the control set $\mathbf{u}$, an upper bound on the total duration of the trajectory $t_{\max}$, and the tolerances ε and δ. The output is a list of (α, r) values describing control line placements. One of these placements is guaranteed to contain an extremal of the correct structure and duration to approximately reach any particular goal configuration closer than $t_{\max}$, with time no worse than the optimal time plus δ.

1. Based on $t_{\max}$, calculate the maximum possible segment count, $n_{\max}$. Periods have a bounded number of segments. In [7], generic extremal trajectories were divided into two types: *roll* and *shuffle* trajectories. It was further shown that shuffle trajectories cannot be longer than one period. For roll trajectories, a period

requires orientation of the robot change 2π, which takes a certain amount of time. Therefore, $t_{\max}$ can be used to bound the number of periods that a roll trajectory can include.

2. Calculate step size Δt. In theorem 3, we proved that for any trajectory, if the total duration of the trajectory is changed by some δ, no point on the trajectory will move more than ε. So, for any given ε, we can calculate a satisfying δ. We can then use theorem 4 to compute a sampling step size Δt, with $\Delta t \leq \delta / n_{\max}$.
3. Calculate every possible extremal trajectory structure given the set of controls, using the approach outlined in [7].
4. For each trajectory structure, find a most sensitive segment k, using the method outlined in the proof of theorem 4.
5. For each trajectory structure, sample the duration, t_k, of the most sensitive segment (using Δt as step size) and compute the corresponding value for H using theorem 1. If segment k is rotation, an upper bound on t_k is $2\pi/\omega_k$; if translation, the upper bound is $t_{\max}$.
6. For each trajectory structure, for each H value, consider each control in the structure as a possible first control. Sample the first control duration t_1 using Δt. Use t_1, H, the identity of the first control, and theorem 2, to compute the location of the control line, (α, r). Save each (α, r) pair to a list.
7. For each computed control line placement, sample the total duration of the trajectory, and use the control line to generate (simulate) the trajectory. For each sampled total duration, calculate the Δq and build the mapping between the trajectory (control line) and Δq.

The algorithm described generates several trajectories that start at a given starting configuration (canonically, the origin), at least one of which is guaranteed to pass within within ε distance of any chosen goal that can be reached in time less than $t_{\max}$, and further, is within δ time of the cost of the minimum-time trajectory. δ and ε may be chosen by the user to be arbitrarily small, subject to available computational resources.

The argument is as follows. Between a particular q_s and q_g, there exists a time-optimal trajectory; call it Y^*. This optimal trajectory must be extremal, has a particular structure S^*, and has particular durations for each segment. The algorithm considers each possible trajectory structure, and must therefore find a trajectory with structure S^*. Further, S^* has a most sensitive segment s_k^*, with duration t_k^*. A sampled trajectory is considered with duration t_k such that $|t_k^* - t_k| < \Delta t$, and because s_k is the most sensitive segment, every other segment duration must be within Δt of the optimal.

6 Implementation results

We implemented a simpler (but equally correct) variation on the algorithm described above in C code. The primary difference between the two algorithms is that instead of looping over trajectory structures, the variation loops instead over possible trajectory substructures $U(s_{k-1})$, $U(s_k)$ and $U(s_{k+1})$. To ensure that all trajectory structures are considered, it is also required to add a set of certain critical values of H to the generated list of H values.

We used the implementation first to test the equations described in theorems 1 and 2 for computing r and α given t_1 and t_k, against known analytical results for the differential robot and the omnidirectional robot that compute all segment durations for a given r and α.

We also used the sample implementation to generate the set of (α, r) values for the asymmetric differential-drive robot drive with one wheel that can drive 1.8 faster than the other, with $\varepsilon = .1$, where the distance between wheels is 2. Figure 5 shows an example near-optimal trajectory. On a standard desktop machine, generating the complete set of about 10, 000, 000 (α, r) pairs required about nine minutes; using this set to generate (simulate) a trajectory for each (α, r) pair (with $t_{\max} = 10$) required a further nine minutes. Though computationally expensive, it should be pointed out that this solution finds the complete mapping from a starting configuration to all goals reachable within 10 seconds.

Further testing and simulation of other systems are a current goal, as well as techniques to improve precision and run-time.

7 Conclusion

The algorithm we presented is fairly specific to a limited class of mobile robots. However, the approach highlights some ideas that we hope will prove a starting point for work at the intersection of optimal control and the design of planning algorithms. First, strong geometric conditions on optimal trajectories derived analytically may allow representation of optimal trajectories using only a few parameters. Second, searching this space of parameters directly may miss certain trajectory structures. However, an indirect sampling method may allow construction of a set of sampled parameters that do allow guarantees on optimality.

The algorithm presented, in order to guarantee near-optimality, samples trajectories to all configurations in some region – an "all-pairs" approach to the motion planning problem. Once constructed, this mapping may allow an analysis of the structure of optimal trajectories of the configuration space, and may be useful for fast queries about trajectories between particular pairs of configurations.

A related interesting problem that we have not considered in the present paper is finding a trajectory between a particular pair of configurations, more precisely and efficiently than finding the complete mapping. Theoretical results about the relationship between parameters that describe trajectories and the duration of trajectory segments may allow more sophisticated branch-and-bound or local-search algorithms that are still provably good approximations.

This work was supported in part by NSF CAREER grant IIS-0643476; we also thank Andrei Furtuna for his advice and suggestions.

References

1. Devin J. Balkcom, Paritosh A. Kavathekar, and Matthew T. Mason. Time-optimal trajectories for an omni-directional vehicle. *International Journal of Robotics Research*, 25(10):985–999, 2006.
2. Devin J. Balkcom and Matthew T. Mason. Time optimal trajectories for differential drive vehicles. *International Journal of Robotics Research*, 21(3):199–217, 2002.
3. Jérôme Barraquand and Jean-Claude Latombe. Nonholonomic multibody mobile robots: Controllability and motion planning in the presence of obstacles. In *IEEE International Conference on Robotics and Automation*, pages 2328–2335, Sacramento, CA, 1991.
4. L. E. Dubins. On curves of minimal length with a constraint on average curvature and with prescribed initial and terminal positions and tangents. *American Journal of Mathematics*, 79:497–516, 1957.
5. C. Fernandes, L. Gurvits, and Z.X. Li. A variational approach to optimal nonholonomic motion planning. In *IEEE International Conference on Robotics and Automation*, pages 680–685, April 1991.
6. Andrei A. Furtuna. Minimum time kinematic trajectories for self-propelled rigid bodies in the unobstructed plane. Dartmouth PhD thesis, Dartmouth College, 2004.
7. Andrei A. Furtuna and Devin J. Balkcom. Generalizing Dubins curves: Minimum-time sequences of body-fixed rotations and translations in the plane. *International Journal of Robotics Research*, 29(6):703–726, 2010.
8. Andrei A. Furtuna, Wenyu Lu, Weifu Wang, and Devin J. Balkcom. Minimum-time trajectories for kinematic mobile robots and other planar rigid bodies with finite control sets. In *IROS*, pages 4321–4328, 2011.
9. Jean-Bernard Hayet, Claudia Esteves, and Rafael Murrieta-Cid. A motion planner for maintaining landmark visibility with a differential drive robot. In *Workshop on the Algorithmic Foundations of Robotics*, December 2008.
10. L.E. Kavraki, P. Svestka, J.-C. Latombe, and M.H. Overmars. Probabilistic roadmaps for path planning in high-dimensional configuration spaces. *Robotics and Automation, IEEE Transactions on*, 12(4):566 –580, aug 1996.
11. Jean-Paul Laumond. Feasible trajectories for mobile robots with kinematic and environment constraints. In *International Conference on Intelligent Autonomous Systems*, pages 346–354, 1986.
12. Steven M. LaValle. *Planning Algorithms*. Cambridge Press, 2006. Also freely available online at http://planning.cs.uiuc.edu.
13. Kevin M. Lynch. The mechanics of fine manipulation by pushing. In *IEEE International Conference on Robotics and Automation*, pages 2269–2276, Nice, France, 1992.
14. H.J Oberle and W. Grimm. BNDSCO: a program for the numerical solution of optimal control problems. 1989.
15. Nathan Ratliff, Matthew Zucker, J. Andrew (Drew) Bagnell, and Siddhartha Srinivasa. CHOMP: Gradient optimization techniques for efficient motion planning. In *IEEE International Conference on Robotics and Automation (ICRA)*, May 2009.
16. J. A. Reeds and L. A. Shepp. Optimal paths for a car that goes both forwards and backwards. *Pacific Journal of Mathematics*, 145(2):367–393, 1990.
17. P. Souères and J.-D. Boissonnat. Optimal trajectories for nonholonomic mobile robots. In J.-P. Laumond, editor, *Robot Motion Planning and Control*, pages 93–170. Springer, 1998.
18. Héctor Sussmann and Guoqing Tang. Shortest paths for the Reeds-Shepp car: a worked out example of the use of geometric techniques in nonlinear optimal control. SYCON 91-10, Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, 1991.
19. M. Vendittelli, J.P. Laumond, and C. Nissoux. Obstacle distance for car-like robots. *IEEE Transactions on Robotics and Automation*, 15(4):678–691, 1999.
20. Weifu Wang and Devin J. Balkcom. Analytical time-optimal trajectories for an omni-directional vehicle. In *IEEE International Conference on Robotics and Automation*, 2012.